

# Acoustic cloaking in 2D and 3D using finite mass

Andrew N. Norris\*

*Mechanical and Aerospace Engineering, Rutgers University, Piscataway NJ 08854<sup>†</sup>*

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Fundamental features of rotationally symmetric acoustic cloaks with anisotropic inertia are derived. Two universal relations are found to connect the radial and transverse phase speeds and the bulk modulus in the cloak. Perfect cloaking occurs only if the radial component of the density becomes infinite at the cloak inner boundary, requiring an infinitely massive cloak. A practical cloak of finite mass is defined in terms of its effective visible radius, which vanishes for perfect cloaking. Significant cloaking is obtained when the effective visible radius is subwavelength, reducing the total scattering cross section, and may be achieved even as the interior radius of the cloak is large relative to the wavelength. Both 2D vs. 3D effects are compared as we illustrate how the spatial dependence of the cloaking parameters effect the total cross section.

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The fundamental observations of Pendry et al. [1] and of Leonhardt [2] that the electromagnetic equations remain invariant under spatial transformations has generated significant interest in the possibility of passive acoustic cloaking. The idea is to map a region around a single point in such a way that the mapping is one-to-one everywhere except at the point, which is mapped into the cloak inner boundary.

Consequences of the transformation method for EM cloaking have been studied extensively. Thus, Rahm et al. [3] demonstrate square cloaks; Chen and Chan [4] provide an interesting mapping that rotates the EM field inside the cloaking region; Chen et al. [5] examine in detail the interaction with a spherical cloak characterized by the linear radial transformation [1]; Zhang et al. [6] consider cloaking in inhomogeneous environments; Huang et al. [7] describe a method to realize cylindrical cloaking using a structure with “normal” EM layers; Kwon and Werner [8] extend the linear transformation to an eccentric elliptic annular geometry lacking rotational symmetry; a quadratic cloak was proposed by and investigated by Yan et al. [9]; Cai et al. [10] use a similar quadratic spatial transformation to obtain a smooth transition in the moduli at the outer interface; and Leonhardt and Philbin [11] provide a general theory of the transformation method in the context of EM waves.

An acoustic fluid is described by a bulk modulus  $K$  and density  $\rho$ , but cloaking cannot occur if the bulk modulus and density are scalar quantities. It is possible to obtain acoustical cloaks by assuming a two parameter density tensor [12, 13, 14]. A tensorial density is not ruled out on basic principles [15] and may in fact be realized through so-called metamaterials. The general context for anisotropic inertia is the Willis equations of elastodynamics [16] which Milton et al. [15] showed are the natural counterparts to the EM equations that remain invariant

under spatial transformation. Acoustic cloaking has been demonstrated, theoretically at least, in both 2D and 3D: a spherically symmetric cloak was discussed by Chen and Chan [13] and by Cummer et al. [12], while Cummer and Schurig [14] described a 2D cylindrically symmetric acoustic cloak. These papers use a linear transformation based on prior EM results in 2D [17]. While the focus here is on passive cloaking, we note that Miller [18] describes possible active acoustic cloaking strategies.

The 2- and 3D cloaks of [12, 13, 14] are shown here to be special cases of a more general class of acoustic cloaks. We will show that the linear transformation proposed in [1] and examined by [5, 12] is just a special case of a general class of transformations possible in acoustic cloaking. The arbitrary nature of the spatial transformation function has been noted [10] but not utilized much as yet, except for quadratic cloaks [9, 10]. In this paper we show that the cloak density tensor depends upon the arbitrary spatial mapping function. Issues that have not been previously considered, such as the total amount of mass required for cloaking, are addressed. It turns out that perfect cloaking occurs only if infinite mass is available, regardless of the spatial mapping function. We take the point of view that achieving infinite mass is impractical, but that effective cloaking with finite mass is still possible if the cloak parameters are appropriately chosen. We begin with a derivation of the general form of the density tensor and bulk modulus necessary for acoustic cloaking with rotational symmetry in both 2D and 3D.

The equations governing small amplitude disturbances in an acoustic metamaterial are

$$\rho \dot{\mathbf{v}} = -\text{grad } p, \quad \dot{p} = -K \text{div } \mathbf{v}. \quad (1)$$

Here,  $p(\mathbf{x}, t)$  and  $\mathbf{v}(\mathbf{x}, t)$  are the pressure and particle velocity field variables, and  $K(\mathbf{x}) > 0$  is the bulk modulus. The density,  $\rho(\mathbf{x})$ , is a symmetric second order tensor. Traditional acoustics corresponds of course to the diagonal form  $\rho = \rho \mathbf{I}$ . Equations (1) imply that the pressure satisfies the generalized acoustic wave equation

$$K \text{div} (\rho^{-1} \text{grad } p) - \ddot{p} = 0. \quad (2)$$

\*Electronic address: norris@rutgers.edu

<sup>†</sup>URL: <http://www.mechanical.rutgers.edu/norris>

We examine the 2D and 3D configurations in parallel.

Consider spatially inhomogeneous anisotropic materials that are locally transversely isotropic, that is, characterized by an axis of symmetry and an orthogonal hyperplane of isotropy. The axis of symmetry is in the radial direction, such that the density matrix has the form

$$\boldsymbol{\rho} = \rho_r(r)\hat{\mathbf{x}} \otimes \hat{\mathbf{x}} + \rho_\perp(r)(\mathbf{I} - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}}), \quad (3)$$

where  $r = |\mathbf{x}|$  and  $\hat{\mathbf{x}} = \mathbf{x}/r$ . The bulk modulus, being a scalar, depends only on radius,  $K(r)$ . The governing equation for  $p$  is

$$\frac{K(r)}{r^{d-1}} \frac{\partial}{\partial r} \left( \frac{r^{d-1}}{\rho_r(r)} \frac{\partial p}{\partial r} \right) + \frac{K(r)}{r^2 \rho_\perp(r)} \Delta_\perp p - \ddot{p} = 0, \quad (4)$$

where  $d$  is the dimension (2 or 3) and  $\Delta_\perp p$  is the Beltrami-Michel operator.

The central idea is to contract or shrink the radial coordinate  $r \rightarrow f(r)$ , with the pressure parameterized in terms of the contracted radial coordinate as  $p(\mathbf{x}, t) = P(f(r), \hat{\mathbf{x}}, t)$ . The scalar governing equation (4) becomes

$$f'(r) \frac{K(r)}{r^{d-1}} \frac{\partial}{\partial f} \left( \frac{r^{d-1} f'}{\rho_r(r)} \frac{\partial P}{\partial f} \right) + \frac{K(r)}{r^2 \rho_\perp} \Delta_\perp P - \ddot{P} = 0, \quad (5)$$

where  $f' = df/dr$ . At the same time, the scalar wave equation is  $\nabla_f^2 P - \ddot{P} = 0$  in the coordinates  $(f, \hat{\mathbf{x}})$  where  $\nabla_f^2$  is the Laplacian in these un-contracted coordinates. The pressure simultaneously satisfies this uniform wave equation in  $(f(r), \hat{\mathbf{x}})$  and eq. (5) in  $(r, \hat{\mathbf{x}})$  if and only if the following three conditions are met:

$$\rho_r = \frac{r^{d-1}}{f^{d-1}} f', \quad \rho_\perp = \frac{r^{d-3}}{f^{d-3} f'}, \quad K = \frac{r^{d-1}}{f^{d-1} f'}. \quad (6)$$

The exterior of  $r = b$  is a uniform and isotropic acoustic medium with  $K = 1$ ,  $\boldsymbol{\rho} = \mathbf{I}$ . Continuity at  $r = b$  requires matching of the pressure and the normal (radial) velocity, and hence the radial acceleration. The latter follows from eq. (1) as  $\dot{v}_r = -\rho_r^{-1} \partial p / \partial r$ , or  $\dot{v}_r = -(f/r) f^{d-1} \partial P / \partial f$ . Hence, all that is required for continuity at  $r = b$  is that  $f$  be continuous across the boundary, which is accomplished by requiring  $f(b) = b$ . Surprisingly, the two conditions at  $r = b$  reduce to this single requirement.

Equations (6) provide relations for the anisotropic acoustic material properties for a given contraction  $f(r)$ . It is useful to express them in terms of the radial and azimuthal phase speeds:  $c_r = \sqrt{K/\rho_r}$  and  $c_\perp = \sqrt{K/\rho_\perp}$ . The mass density tensor can then be expressed  $\boldsymbol{\rho} = K(c_r^{-2} \mathbf{I}_r + c_\perp^{-2} \mathbf{I}_\perp)$ . Equations (6) imply the identity

$$K^{d-2} = \rho_r \rho_\perp^{d-1}, \quad (7)$$

independent of the choice of contraction function  $f(r)$ . The quantity  $K\rho_r$  is the square of the radial acoustic impedance,  $z_r \equiv \sqrt{K\rho_r}$ . Equation (7) implies that the identity  $z_r = c_\perp^{d-1}$  is required for cloaking. The three

equations (6) can be replaced by (7) along with relations for the phase speeds in terms of  $f$ :

$$c_r = 1/f', \quad c_\perp = r/f. \quad (8)$$

Note that  $f'$  is required to be positive. The original quantities can be expressed in terms of the phase speeds as

$$\rho_r = c_r^{-1} c_\perp^{d-1}, \quad \rho_\perp = c_r c_\perp^{d-3}, \quad K = c_r c_\perp^{d-1}. \quad (9)$$

One could, for instance, eliminate  $f$  as the fundamental variable defining the cloak in favor of  $c_\perp(r)$ , from which all other quantities can be determined from the differential equation relating the speeds:  $(r/c_\perp)' = 1/c_r$ . We assume the cloak occupies  $\Omega = \{0 < a \leq r \leq b\}$  with uniform acoustical properties  $K = 1$ ,  $\boldsymbol{\rho} = \mathbf{I}$  in the exterior. Then  $c_\perp(r)$  follows by integration,

$$\frac{1}{c_\perp} = \frac{b}{r} - \frac{1}{r} \int_r^b \frac{dr}{c_r}, \quad a \leq r \leq b. \quad (10)$$

Can the cloak density be isotropic? This will occur if  $c_r = c_\perp$ , which requires that  $f' = f/r$ , implying  $f = \gamma r$  with  $\gamma$  constant. Impedance matching at the cloak outer boundary  $r = b$  requires  $f(b) = b$ . This can only be satisfied in the trivial case of  $\gamma = 1$ , implying  $f \equiv r$ . While this eliminates the possibility of isotropic cloaks, it is perhaps a satisfying outcome.

Perfect cloaking requires that  $f$  vanish for some  $r = a_0 > 0$ . It is clear that  $c_\perp$  and  $z_r$  blow up as  $r \downarrow a_0$ . Hence the product  $K\rho_r$  blows up, but do  $K$  and  $\rho_r$  each become unbounded? Consider  $f \propto (r - a_0)^\alpha$  near  $a_0$  for  $\alpha$  constant and non-negative. No value of  $\alpha > 0$  will keep the radial density  $\rho_r$  bounded, but the unique choice  $\alpha = 1/d$  ensures that the bulk modulus  $K(a_0)$  remains finite. We will see below that this power law is not a

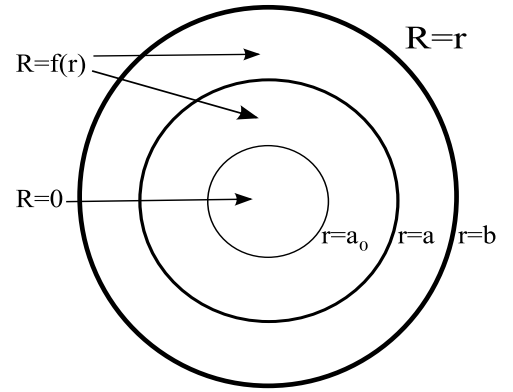


FIG. 1: The cloak occupies  $a < r < b$  with uniform properties outside and the “hidden” object inside. The cloak properties depend upon a monotone contraction mapping  $f(r)$  within the cloak, such that  $f(b) = b$ ,  $f(a) < a$ . It is useful to think of an invisibility boundary  $r = a_0$  inside the cloak, where  $f(a_0) = 0$ . The pressure satisfies a uniform wave equation in the mapped radial variable  $R(r)$ .

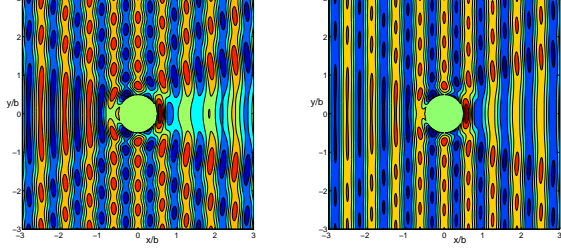
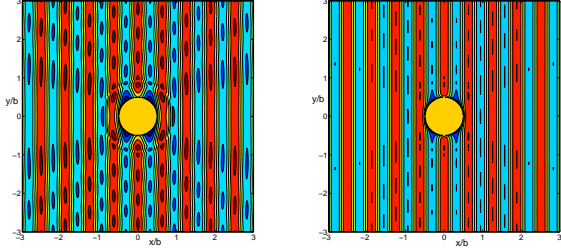
(a) 2D,  $\alpha = 1$ ,  $\sigma_{tot} = 1.18$ (b) 3D,  $\alpha = 1$ ,  $\sigma_{tot} = 0.49$ (c) 2D,  $\alpha = 4$ ,  $\sigma_{tot} = 0.06$ (d) 3D,  $\alpha = 4$ ,  $\sigma_{tot} = 10^{-4}$ 

FIG. 2: A plane wave is incident from the left with frequency  $k = 10$  on the cloak defined by eq. (13) with  $(b, a, a_0) = (1, 0.5, 0.35)$ . The exponent is  $\alpha = 1$  in (a) and (b) implying a virtual inner radius of  $f(a) = 0.23$ . For (c) and (d),  $\alpha = 4$  giving  $f(a) = 0.003$ .

TABLE I: Behavior of quantities near the vanishing point  $r = a_0$  for the scaling  $f \propto x^\alpha$  as  $x = r - a_0 \downarrow 0$ . The total radial mass  $m_r$  is defined in eq. (11).

dim	$\rho_r$	$\rho_\perp$	$c_r$	$c_\perp$	$K$	$m_r$
2	$x^{-1}$	$x$	$x^{1-\alpha}$	$x^{-\alpha}$	$x^{1-2\alpha}$	$\ln x$
3	$x^{-1-\alpha}$	$x^{1-\alpha}$	$x^{1-\alpha}$	$x^{-\alpha}$	$x^{1-3\alpha}$	$x^{-\alpha}$

practical choice for the cloak mapping function. Note that the azimuthal density  $\rho_\perp$  has a finite limit in 2D for power law decay  $f \propto (r - a_0)^\alpha$ , while  $\rho_\perp$  remains finite in 3D iff  $\alpha \leq 1$ , otherwise it blows up. Similarly, the radial phase speed scales as  $c_r \propto (r - a_0)^{1-\alpha}$ , which remains

finite for  $\alpha \leq 1$ , blowing up otherwise. These results are summarized in Table I.

We introduce a nondimensional measure of the total amount of cloaking mass present in the radial component of the inertia tensor:

$$m_r \equiv \frac{\int_\Omega dV \rho_r}{\int_\Omega dV} = \frac{d}{b^d - a^d} \int_a^b dr r^{2(d-1)} \frac{f'(r)}{f^{d-1}(r)}. \quad (11)$$

Explicitly,

$$m_r = \begin{cases} \frac{2}{b^2 - a^2} [b^2 \ln f(b) - a^2 \ln f(a) - 2 \int_a^b dr r \ln f(r)], \\ \frac{3}{b^3 - a^3} [\frac{a^4}{f(a)} - \frac{b^4}{f(b)} + 4 \int_a^b dr r \frac{r^3}{f(r)}], \end{cases} \quad (12)$$

in 2D and 3D, respectively. These forms indicate not only that  $m_r \rightarrow \infty$  as  $f(a) \rightarrow 0$ , but also the form of the blow-up. To leading order,  $m_r = \frac{2a^2}{b^2 - a^2} \ln \frac{1}{f(a)} + \dots$  in 2D and  $m_r = \frac{3a^3}{b^3 - a^3} \frac{a}{f(a)} + \dots$  in 3D. The blow-up of  $m_r$  occurs no matter how  $f$  tends to zero; that is, the infinite mass is an unavoidable singularity.

The idea behind the contraction function  $f(r)$  is to map the entire interior of  $r \leq b$  into an annular subset:  $\Omega$ . This in turn implies that perfect cloaking requires infinite mass  $m_r$ . We take the point of view that  $m_r$  must remain finite, which can be achieved by making  $f > 0$  in  $\Omega$ . To be specific, consider the analytic continuation of  $f(r)$  into the “cloak”  $\{r : 0 \leq r < a\}$ , where we allow  $f(r)$  to vanish. Consider the power law form for the mapping function,  $f(r) = f^\alpha(r)$  where

$$f^{(\alpha)}(r) = b \left( \frac{r - a_0}{b - a_0} \right)^\alpha, \quad (13)$$

for  $0 \leq a_0 < a$ . This yields finite mass  $m_r$ , but it blows up as  $a \downarrow a_0$  for every  $\alpha > 0$ .

It follows from (12) that the dependence of the total mass on  $\alpha$  is linear in 2D:  $m_r^{(\alpha)} = \alpha m_r^{(1)}$ . In other words, in 2D the mass required is proportional to the log of the effective vanishing radius. For 3D, the leading order term in eq. (12) implies  $m_r^{(\alpha)} \approx \beta^{-1} (\beta m_r^{(1)})^\alpha$ , where  $\beta = \frac{1}{3} \frac{b}{a} (\frac{b^3}{a^3} - 1)$ . The total mass grows exponentially with  $\alpha$ , all other parameters being fixed. However, we can still achieve effective cloaking with finite  $m_r$ . The idea is to make  $f(a)$  subwavelength. This does *not* require that  $a$  be subwavelength.

We assume time harmonic motion, with the factor  $e^{-ikt}$  understood but omitted. Consider a cloak of finite mass with zero pressure on the interior surface  $r = a$ . The total response for plane wave incidence is

$$p = p_0 e^{ikR(r) \cos \theta} - p_0 \sum_{n=0}^{\infty} \begin{cases} i^n (2 - \delta_{n0}) J_n(kR(a)) \frac{H_n^{(1)}(kR(r))}{H_n^{(1)}(kR(a))} \cos n\theta, & 2D, \\ i^n (2n+1) j_n(kR(a)) \frac{h_n^{(1)}(kR(r))}{h_n^{(1)}(kR(a))} P_n(\cos \theta), & 3D, \end{cases} \quad R(r) = \begin{cases} f(r), & a \leq r < b, \\ r, & b \leq r < \infty. \end{cases} \quad (14)$$

The total scattering cross-section, and hence the total energy scattered, depends on the forward scattering amplitude through the optical theorem:  $\sigma_{tot} = 4\pi k^{-1} \text{Im} g(\hat{\mathbf{e}}_z)$ , where  $g$  defines the leading order far-field  $p = p_0 e^{ikz} + p_0 g(\hat{\mathbf{x}}) r^{-(d-1)/2} (i2\pi/k)^{(3-d)/2} e^{ikr}$ . The scattering cross-section is easily computed from (14), and is dominated in the small  $kR(a)$  limit by the  $n = 0$  term, with leading order approximations

$$\sigma_{tot} = \begin{cases} \frac{\pi^2}{k} |\ln kf(a)|^{-2} + \dots, & 2D, \\ 4\pi f^2(a) + \dots, & 3D. \end{cases} \quad (15)$$

The faster decay of  $\sigma_{tot}$  explains the greater efficacy of the finite mass cloak in 3D. Considered as a function of the power  $\alpha$ , in 3D the mass  $m_r$  *increases* exponentially while the cross-section  $\sigma_{tot}$  *decreases* exponentially. In 2D the rates of increase and decrease of  $m_r$  and  $\sigma_{tot}$  are algebraic (linear) or less. The comparison suggests that cylindrical cloaks require that the visibility radius  $f(a)$  be very small. Figure 2 illustrates clearly the disparity between cylindrical and spherical cloaking. Thus, for  $f(a)/b = 0.003$  the 3D cross-section is negligible (Fig. 2.d)

but for 2D the cross-section is two orders of magnitude larger (Fig. 2.c). Note that Ruan et al. [19] found that the perfect cylindrical EM cloak is sensitive to perturbation. This sensitivity is evident from the present analysis through the dependence on the length  $a - a_0 > 0$  which measures the departure from perfect cloaking ( $a = a_0$ ).

In conclusion, we have shown how to generate acoustic cloaks in 2D and 3D using a general radial transformation  $r \rightarrow f(r)$ . Regardless of the form of  $f$ , two universal equations relate the cloak parameters, e.g.  $K = c_r c_\perp^{d-1}$  and  $(r/c_\perp)' = 1/c_r$ . Perfect cloaking requires not only that the density  $\rho_r \rightarrow \infty$  at the inner radius, but the associated total mass  $m_r$  is also infinite. This can be avoided in practice since the scattering cross-section  $\sigma_{tot}$  can be made as small as desired by selecting the value of  $f(a) \ll a$  so that  $kf(a) \ll 1$  even as  $ka = O(1)$ . We showed that the form of the cloak mapping function  $f(r)$  is important and the effect of the power  $\alpha$  is markedly different in 2D vs. 3D. The feasibility of cloaking depends on the development of real metamaterials with the desired unusual physical properties. This is an active research area [20, 21, 22] with tremendous potential for future applications.

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